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Longitudinal size exponent for square-lattice directed animals

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Abstract. Series expansions for the longitudinal gyration radius of square-lattice directed-site animals to order $N = 39$ are given. This doubles the length of the available series. Analysis of the series yields the estimate $\nu_{\parallel} = 0.817\,22 \pm 0.000\,05$, which excludes the conjecture $\nu_{\parallel} = \frac{9}{11}$, based on phenomenological renormalization and shorter series, but is consistent with the later phenomenological renormalization estimate of Dhar: $\nu_{\parallel} = 0.817\,33 \pm 0.000\,05$.

1. Introduction

The square-lattice directed-animals problem is curious in that it shows both simple and complex critical behaviour within the same model. The number of distinct animals with n sites as well as their average width is known exactly [1, 2] but the longitudinal size exponent $\nu_{\parallel}$, defined by $\langle R_{\parallel}^2 \rangle_n \sim n^{2\nu_{\parallel}}$, is not known exactly. Directed-site percolation can also be readily formulated in terms of these same animals, and yet there is no exact knowledge of percolation probabilities or exponents for this model. The generating function $f_1(x)$ for the number of directed animals of n-sites satisfies the quadratic equation

$$(3x-1)f_1^2 + (3x-1)f_1 + x = 0.$$

Other properties satisfy a similar equation. The perimeter-generating function $f_2(x)$, whose coefficients are the number of perimeter sites of all n-site directed animals, satisfies

$$x(x+1)^3(3x-1)^3 f_2^2 + (x^2+x+1)(x+1)^2(3x-1)^3 f_2 + x(2-6x-5x^2+12x^3+13x^4+12x^5+9x^6) = 0.$$

The generating function for the second moment of width $f_3(x)$, whose coefficients are the total moment of inertia around the preferred direction, satisfies

$$(x+1)(3x-1)^5 f_3^2 + 4x^4 = 0.$$

The generating function for the number of loops $f_4(x)$, whose coefficients are the total number of loops in all directed-n-site animals, satisfies

$$(x+1)(3x-1)^3 f_4^2 + (x+1)(3x-1)^3 f_4 + x^4(4x^2+2x-1) = 0.$$

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Table 1. The sum over all animals of longitudinal radius of gyration. The premultiplicative factor makes it an integer. c_n is the number of animals.

n	c_n	$n^2 c_n R_n^2$
0	0	0
1	1	0
2	2	2
3	5	26
4	13	206
5	35	1290
6	96	7008
7	267	34 616
8	750	159 620
9	2123	698 784
10	6046	2 937 078
11	17 303	11 947 070
12	49 721	47 304 690
13	143 365	183 126 678
14	414 584	695 469 488
15	1 201 917	2 598 054 568
16	3 492 117	9 567 485 872
17	10 165 779	34 792 851 512
18	29 643 870	125 128 314 886
19	86 574 831	445 576 045 958
20	253 188 111	1 572 657 087 402
21	741 365 049	5 506 449 749 094
22	2 173 243 128	19 140 831 550 468
23	6 377 181 825	66 097 278 935 264
24	18 730 782 252	226 874 601 290 480
25	55 062 586 341	774 431 863 828 992
26	161 995 031 226	2 630 059 405 421 818
27	476 941 691 177	8 889 989 064 525 018
28	1 405 155 255 055	29 918 515 974 127 482
29	4 142 457 992 363	100 280 380 960 533 778
30	12 219 350 698 880	334 849 192 296 398 760
31	36 064 309 311 811	1 114 161 731 736 449 424
32	106 495 542 464 222	3 694 969 779 198 229 800
33	314 626 865 716 275	12 215 917 498 976 013 160
34	929 947 027 802 118	40 269 358 644 255 614 746
35	2 749 838 618 630 271	132 382 494 758 429 251 502
36	8 134 527 149 366 543	434 071 400 548 028 113 250
37	24 072 650 378 629 801	1 419 804 361 024 411 564 838
38	71 264 483 181 775 040	4 633 290 589 211 722 421 468
39	211 043 432 825 804 129	15 086 788 088 213 710 648 984

The generating function for the square of the number of loops $f_5(x)$, whose coefficients are the square of the number of loops in all directed-n-site animals, satisfies

$$(x+1)^3(3x-1)^5 f_5^2 + (x-1)(2x^2+4x+1)(x+1)^2(3x-1)^5 f_5 + x^4(-1+4x+10x^2-52x^3+27x^4+48x^5-108x^6+144x^7+252x^8) = 0.$$

All these equations are obtained in [1].

The earliest study of $\nu_\parallel$ appears to be by Nadal *et al* [3] who used a phenomenological renormalization group and estimated $\nu_\parallel = 0.8185 \pm 0.0010$ and noted that this was close to the simple fraction $\frac{9}{11} = 0.818\,181\ldots$. Later, Privman and Barma [4] obtained series to order $N = 19$ and from their analysis obtained $\nu_\| = 0.8177 \pm 0.0012$, in agreement with the earlier estimate. Later, Dhar [5] estimated $\nu_\|$ by a phenomenological renormalization with strip widths up to 23 and obtained the much more precise estimate $\nu_\| = 0.817\,33 \pm 0.000\,05$, which appears to rule out the earlier conjectured value.

In this paper, we have more than doubled the series and a careful analysis confirms the results of Dhar and, in particular, excludes the earlier conjecture $\frac{9}{11}$. It should also be noted that Dhar defines the average width to be the average maximal span, whereas we use the average of the sum of the squares of the distance of each site from the centre line of the animal.

The series have been generated by a transfer-matrix method [6] and the coefficients are given in table 1.

2. Analysis

We have analysed the series by a variety of methods. The 'generating function' of the gyration radius is singular at 1, with exponent $1 + 2\nu_\|$. Differential approximants [7] are all found to be defective, so any conclusions drawn therefrom need to be regarded with caution. They also show a decreasing trend with order, from which we can only estimate $\nu_\| < 0.8178$, which is, however, sufficient to rule out the conjecture $\frac{9}{11}$.

However, the series is sufficiently well behaved such that a variety of standard sequence extrapolation algorithms can be directly applied. In particular, if $R_n^2 \sim n^{2\nu}[A + B/n^\Delta]$ then $(R_n^2/R_{n-1}^2 - 1)n \sim 2\nu + \mathrm{O}(1/n) + \mathrm{O}(1/n^\Delta)$. A variety of extrapolation procedures [7], including Brezinski's θ algorithm, Levin's u-transform and Neville extrapolation, all give estimate sequences of 2ν that are decreasing, allowing us, once again, to only estimate $\nu_\| < 0.8178$. However, if we assume the existence of an analytic term in $\langle R_\|^2 \rangle_n$, so that

$$\langle R_\|^2 \rangle_n \sim An^{2\nu} + Bn + \cdots$$

it then follows that $(\langle R_\|^2 \rangle_n / \langle R_\|^2 \rangle_{n-1} - 1)n \sim 2\nu + \mathrm{O}(n^{1-2\nu})$, or that the leading correction-to-scaling exponent $\Delta = 2\nu - 1$. Taking $\nu \approx 0.818$ then implies $\Delta \approx 0.636$. We therefore fitted the sequence $(\langle R_\|^2 \rangle_n / \langle R_\|^2 \rangle_{n-1} - 1)n$ to the assumed form $2\nu + a_1/n^{0.636} + a_2/n + a_3/n^{1.636}$ with the following results:

$$\nu_\| = 0.817\,22 \pm 0.000\,05 \qquad a_1 = 0.1475 \pm 0.002$$

$$a_2 = 0.660 \pm 0.02 \qquad a_3 = -0.290 \pm 0.02.$$

Our estimate of $\nu_\|$ is just in agreement with that of Dhar, our respective estimates and error bounds just meeting at $\nu_\| = 0.817\,27$. If one assumes that the exact value is given by a rational fraction, there are only three possible fractions with denominators less than 200. These are

$$\frac{76}{93} = 0.817\,204\ldots \qquad \frac{161}{197} = 0.817\,258\ldots \qquad \frac{85}{104} = 0.817\,308\ldots.$$

None of these fractions is particularly compelling. A remote possibility is that there are confluent logarithmic terms, in which case our analysis, and indeed all other analyses, cannot be relied upon. However, we consider this unlikely as there are no confluent logarithms in the other exponents for this problem, nor any suggestion that the critical dimension for directed percolation is two.

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